Renormalization-group approach to an Abelian sandpile model on planar lattices

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One important step in the renormalization-group (RG) approach to a lattice sandpile model is the exact enumeration of all possible toppling processes of sandpile dynamics inside a cell for RG transformations. Here we propose a computer algorithm to carry out such exact enumeration for cells of planar lattices in the RG approach to the Bak-Tang-Wiesenfeld sandpile model [Phys. Rev. Lett. **59**, 381 (1987)] and consider both the reduced-high RG equations proposed by Pietronero, Vespignani, and Zapperi (PVZ) [Phys. Rev. Lett. **72**, 1690 (1994)], and the real-height RG equations proposed by Ivashkevich [Phys. Rev. Lett. **76**, 3368 (1996)]. Using this algorithm, we are able to carry out RG transformations more quickly with large cell size, e.g., 3×3 cell for the square (SQ) lattice in PVZ RG equations, which is the largest cell size at the present, and find some mistakes in a previous paper [Phys. Rev. E **51**, 1711 (1995)]. For SQ and plane triangular (PT) lattices, we obtain the only attractive fixed point for each lattice and calculate the avalanche exponent τ and the dynamical exponent z. Our results suggest that the increase of the cell size in the PVZ RG transformation does not lead to more accurate results. The implication of such result is discussed.

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I. INTRODUCTION

In 1987, Bak, Tang, and Wiesenfeld (BTW) [1] proposed the concept of self-organized criticality (SOC) in order to understand the automatic (i.e., without a tuning parameter, such as temperature) appearance of abundant self-similar structures and scaling quantities in nature. BTW also proposed a lattice sandpile model and used Monte Carlo simulations to simulate this model on square and simple cubic lattices. They did observe self-similar structures and scaling quantities in the simulation data without tuning any parameter. Since 1987, many natural phenomena have been related to SOC, such as earthquakes [2], forest fires [3], biological evolution [4], rice pile dynamics [5], turbulence [6], etc. Many lattice models have also been proposed to illustrate the behavior of SOC or avalanche processes [7,8]. It has been found that the BTW's sandpile model is Abelian [9] and some quantities for this model could be calculated exactly [9–12]. The BTW's Abelian sandpile model (ASM) [1] has been considered to be a prototypical model for SOC. Many ideas about behavior of SOC models, such as universality and scaling [13], or methods for studying SOC models, such as renormalization-group theory [14–17], are often tested in the BTW's ASM. In the present paper, we propose a computer algorithm which is useful for carrying out renormalization-group transformations for the BTW's ASM.

The BTW's ASM on a lattice $\Re$ of N sites is defined as follows. Each site of $\Re$ is assigned a height integer; the ith site is assigned z_i for $1\leqslant i\leqslant N$. In the beginning of the simulation, the height integer at each site is randomly chosen to be 0,1, . . . , or z_c-1, where the critical height is z_c-1 and z_c is the coordination number of the lattice. For each time interval T_a, one particle falls on a randomly chosen lattice site, say the ith site; the height z_i then becomes z_i+1. If the new $z_i<z_c$, then randomly choose a site again, say the kth site, to add a particle; if $z_i\geqslant z_c$, then the ith site topples and its height z_i becomes z_i-z_c. At the same time, each of the nearest neighbor (NN) sites of the ith site receives one particle, i.e., $z_{\omega(i,j)}\rightarrow z_{\omega(i,j)}+1$, $\forall\ j$ where $\omega(i,j)$ is the label of the jth NN site of the ith site. This relaxation procedure takes time t_w and we assume $t_w/T_a\rightarrow0$. If some of the new $z_{\omega(i,j)}$, $\forall\ j$, are equal or larger than z_c again, these sites are denoted by $\omega(i,j')$. Then, the toppling process continues in parallel for each j' with $z_{\omega(i,j')}\rightarrow z_{\omega(i,j')}-z_c$ and $z_{\omega(\omega(i,j'),k)}\rightarrow z_{\omega(\omega(i,j'),k)}+1$ where $\omega(\omega(i,j'),k)$ is the kth NN site of the j'th NN site of the ith site. The relaxation time for these parallel toppling and receiving processes between $\omega(i,j')$ and $\omega(\omega(i,j'),k)$ takes another time t_w. Usually, the open boundary conditions are used so that when a boundary site topples, the particle can leave the system. The dynamical process continues until the heights of all sites are less than z_c. In general, if the last toppling site is $\omega(\omega(\cdots(\omega(\omega(i,i_1),i_2),i_3)\cdots),i_n)$, the total toppling process takes time nt_w. In this way, a series of toppling processes with toppling area s (i.e., the total number of toppled sites) and relaxation time $nt_w\equiv t$ appears and forms an avalanche which has no characteristic size.

After repeating many times the process of adding one particle on a randomly chosen site with subsequent relaxation when the height of the site is equal or larger then z_c, we can obtain a distribution of toppling area $P(s)$ and calculate the average relaxation time $\langle t\rangle$ for avalanches with toppling area s. It has been found that $P(s)\sim s^{-\tau}$ with the avalanche exponent τ and $\langle t\rangle\sim s^{z/2}$ with the dynamical exponent z. Manna [18] used Monte Carlo simulations to calculate $\tau=1.22$ and $z=1.21$ for the BTW model on the square (SQ) lattice. Majumdar and Dhar conjectured that $z=5/4$ which is consistent with their own numerical simulations [19], and Priezzhev *et al.* [20] proposed that $\tau=5/4$. By scaling argument, Tebaldi *et al.* [21] suggested $\tau=6/5$. Many investigations have just focused on numerical simulations or

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exact results for height probabilities and exponents for the SQ lattice and there is little attention to other kinds of lattices. It is not clear whether the sandpile model on two-dimensional lattices have the same set of critical exponents.

The renormalization-group (RG) theory has been used successfully to calculate critical exponents, order parameters, etc., of ordinary phase-transition models, e.g., the Ising model [22], the Potts model [23], and the percolation model [24–26]; it has also been used to understand universality and scaling in ordinary phase-transition models. It is of interest to know whether one can also use RG theory to calculate critical exponents for lattice SOC models, e.g., the BTW sandpile model. In 1994, Pietronero, Vespignani, and Zapperi (PVZ) [14] proposed a RG theory to calculate critical exponents of the BTW sandpile model on the SQ lattice. They used a stationary condition to provide a feedback mechanism that drives the system to its critical state. In fact, in their approach, the height of a site z_i is reduced to only three cases corresponding to three classes: stable for $z_i < z_c - 1$, critical for $z_i = z_c - 1$, and unstable for $z_i \geqslant z_c$. The obtained renormalization-group equations allow them to get an attractive fixed point. At this fixed point, they obtained avalanche exponent $\tau = 1.253$ and dynamical exponent $z = 1.168$. In 1995, PVZ [15] described in more detail their RG approach to the BTW sandpile model. In 1996, Ivashkevich [16] generalized PVZ's RG scheme with real height parameters z_i by kinetic equations and used the technique of generating function to construct RG equations to obtain the exponent $\tau = 1.248$ and height probabilities. In addition, Ivashkevich found that for the SQ lattice critical particle transfer probabilities are close to branching probabilities for spanning tree. However, Papoyan and Povolotsky [17] pointed out that such connections could be wrong and used the example of the plane triangular (PT) lattice to illustrate their ideas. In 1996, Vespignani, Zapperi, and Loreto [27] proposed a real-space dynamical-driven renormalization group to provide a theoretical basis for previous RG studies of the ASM. In 1997, Lübeck and Usadel used extensive Monte Carlo simulations to find that $\tau \approx 1.33$ [28].

One important step in the RG approach to a lattice sandpile model is the exact enumeration of all possible toppling processes of sandpile dynamics inside a cell for RG transformation. PVZ and Ivashkevich divided the height configurations inside a RG cell into subsets with the same number of critical height by hand. After this procedure, for each height configuration, PVZ counted the possible toppling events by hand and Ivashkevich did this by the technique of generating function. Based on Refs. [14–16], in the present paper we formulate a computer algorithm for RG studies of the BTW sandpile model on the SQ, PT, and honeycomb (HC) lattices. The advantage of our method is that we do not need to classify all height configurations inside a RG cell into different subsets by hand which is different for different kinds of lattices and RG cell sizes. Therefore, it is easier for us to extend the same algorithm to different cell sizes and different kinds of lattices. Another advantage of our method is that we can study larger RG cells. For example, in [14,16,29] and [30], they have only carried out a 2×2 cell to one site and a five-site cell to one-site RG transformations for the SQ lattice, and we can carry out a 3×3 cell to one-site RG transformation (RGT) for the SQ lattice.

In the present paper, we consider both PVZ and Ivashkevich RGTs. Using the PVZ reduced-height RG equations, we calculate the density of critical sites ρ, the probability for one site to transfer sands to i different nearest neighbors p_i, the avalanche exponent τ and the dynamical exponent z by 2×2 and 3×3 cells to one site RGTs for the SQ lattice, and three sites cell to one-site RGT for the PT lattice. Using the Ivashkevich's real-height RG equations, we calculate the critical height probabilities, the critical sand transfer probabilities, the avalanche exponent τ, and the dynamical exponent z by a 2×2 cell to one-site RGT for the SQ lattice and three-site cell to one-site RGT for the PT lattice. We find that our calculated height probabilities are consistent with the exact and numerical simulations.

The outline of this paper is as follows. In Sec. II, we introduce the reduced-height RG equations. Here, we extend PVZ's formulation for the SQ lattice to a general lattice. In the Appendix, we present a systematic computer algorithm for the exact enumeration of all possible toppling processes of sandpile dynamics inside a cell for RGT. By using this algorithm, we present the obtained fixed points of reduced-height RG equations and the critical exponents in Sec. III. In Sec. IV, we apply real-height RG equations to the BTW model on SQ and PT lattices. For a different definition of the RG cell for the HC lattice, in Sec. V, we calculate the critical sand transfer probabilities and find that they are different from branching probabilities of spanning trees on the same (HC) lattice. This result confirms Papoyan and Povolotsky's result [17] for the PT lattice. Finally, we summarize our results and discuss problems for further studies in Sec. VI.

II. REDUCED-HEIGHT RENORMALIZATION-GROUP EQUATIONS

In this section we extend PVZ's formulation of RG transformation [14] for the SQ lattice to a general lattice $\Re$ and present a simplified version which is useful for realization by a computer program. Then, we propose an algorithm for counting events of RG procedure. In the Appendix, we will show that this calculation is basically consistent with that of Ref. [15] for a 2×2 cell of the square lattice. However, there are some mistakes in Ref. [15].

As in ordinary real-space RG transformations for the spin and percolation models [22–25], in the RG transformation for the BTW sandpile model on the lattice $\Re$ with lattice-constant a, we first divide $\Re \equiv \Re^{(0)}$ into $g\times g$ cells, then consider the transformation from each $g\times g$ cell into a cell with a smaller number of sites or a supersite. In the present paper, we consider only the later case. From the relationship between the cell and the supersite, the properties of the specified cell can be approximately represented by the supersite. After the first step of RG transformation, we have a lattice, called $\Re' \equiv \Re^{(1)}$, of supersites with lattice-constant ag. We can carry out the second step RG transformation on $\Re'$ to obtain a lattice $\Re'' \equiv \Re^{(2)}$ with lattice-constant ag^2 and we can continue such RG transformation to obtain a series of lattices $\Re^{(k)}$ with lattice-constant ag^k. Examples of RG cells

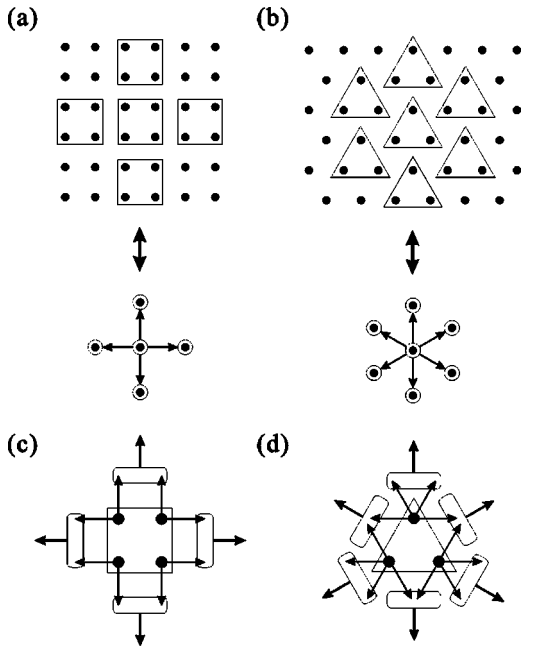

FIG. 1. (a) and (b) Transformation from a cell to a site for the RG transformation. Here, (a) and (b) are for the square lattice with 2×2 cells and for the plane triangular lattice with $\sqrt{3}\times\sqrt{3}$ cells, respectively. Blocks of sites of the coarse-grained lattices $\Re^{(k-1)}$ become supersites on the renormalized lattice $\Re^{(k)}$. (c) and (d) We show that the directions outgoing from the blocks, which are encircled by rectangle corner roundness, are coupled to the directions of the lattice at the next scale.

for SQ and PT lattices are shown in Figs. 1(a) and 1(b), respectively, whose linear dimensions (i.e., g values) are 2 and $\sqrt{3}$, respectively.

Here, we briefly review PVZ's RG approach to the BTW's ASM [14,15]. If we consider a sandpile RG cell with $m=g^2$ sites, the initial height z_i of each site i inside a RG cell can take integer values 0,1, . . . , or $q-1$, where q is the coordination number of lattice $\Re$; for example Figs. 1(a) and 1(b) correspond to the SQ and the PT lattices, with $q=4$ and 6, respectively. After adding one particle to one lattice site, say site i, if the site i is unstable (i.e., $z_i\geqslant z_c$), the site i will topple and transfer particles to e_i different nearest neighbors. In general, e_i can take values 1,2, . . . , or q after RG transformation which is different from $e_i=q$ for the original sandpile evolution on $\Re^{(0)}$. As an illustration, the pictures of e_i for the SQ lattice are shown in Fig. 2. The toppling rule e_i can be described by the transferring probability $\vec{\mathbf{p}}=(p_1,p_2,\ldots,p_q)$ where p_{e_i} is the probability that the unstable site i transfers particles to e_i different nearest neighbors; if e_i is smaller than q, one should also specify which directions the particles go into, which will be explained in more details in the next section. In general, the evolution of the ASM at the initial stage $\Re^{(0)}$, is characterized by the initial height configuration set $Z=\{\vec{z}\}=\{(z_1,z_2,\ldots,z_m)|\ z_i=0,1,\ldots,\text{ or } q-1,\forall\ i\}$ and the toppling rule $\vec{e}=(e_1,e_2,\ldots,e_m)=(q,q,\ldots,q)$. That is $p_1=p_2=\cdots=p_{q-1}=0$ and $p_q=1$ for $\vec{\mathbf{p}}$ at this stage. For example, for

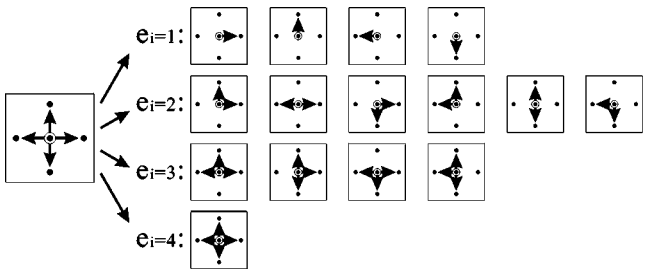

FIG. 2. The picture of toppling rules for a site of the square lattice. The e_i denotes one unstable site i will transfer e_i particles to e_i different nearest neighbors. For each e_i, there are $C_{e_i}^q$ possible toppling rules.

the ASM model on the original square lattice, the height and transferring probability can take value $z_i=0,1,2$, or 3 and $\vec{\mathbf{p}}=(0,0,0,1)$, respectively. On the other hand, after the RG transformation, the height configuration set and toppling rule configuration set are Z and E, respectively, where $E=\{\vec{e}\}=\{(e_1,e_2,\ldots,e_m)|\ e_i=1,2,\ldots,\text{ or } q,\forall\ i\}$.

In this way, the transferring probability $\vec{\mathbf{p}}$ and the initial height probability $\vec{\mathbf{n}}=(n_0,n_1,\ldots,n_{q-1})$, where n_i is the probability that the height of the supersite is i, are useful to characterize the properties of the coarse-grained sandpile dynamics. We use $\vec{\mathbf{p}}^{(k)}=(p_1^{(k)},p_2^{(k)},\ldots,p_q^{(k)})$ and $\vec{\mathbf{n}}^{(k)}=(n_0^{(k)},n_1^{(k)},\ldots,n_{q-1}^{(k)})$ to denote the transferring probability and initial height probability on lattice $\Re^{(k)}$. $(\vec{\mathbf{n}}^{(k)},\vec{\mathbf{p}}^{(k)})$ and $(\vec{\mathbf{n}}^{(k-1)},\vec{\mathbf{p}}^{(k-1)})$ are linked by the kth RG transformation. Instead of $p_i^{(0)}=0$ for $i<q$ and $p_q^{(0)}=1$ and $\Sigma_{i=0}^{q-1}n_i^{(0)}=1$ in the original level, the normalized condition $\Sigma_{i=1}^{q}p_i^{(k)}=1$ and $\Sigma_{i=0}^{q-1}n_i^{(k)}=1$ are used in the coarse-grained level.

In order to simplify the RG transformation, PVZ reduced the height of a site z_i to only three cases corresponding to three classes: $h_i=0$ which is stable for $z_i<z_c-1$, $h_i=1$ which is critical for $z_i=z_c-1$, and $h_i=2$ which is unstable for $z_i\geqslant z_c$. Therefore, Z will be replaced by $H=\{\vec{h}\}=\{(h_1,h_2,\ldots,h_m)|\ h_i=0\ \text{ or }\ 1,\forall\ i\}$. The evolution rule is as follows: If a critical site i with height $h_i=1$ receives one particle, its height is now $h_i=2$ and site i becomes unstable. Then, this unstable site will transfer particles to neighboring sites with the transferring probability $\vec{\mathbf{p}}$. After this action, this toppled site becomes stable with $h_i=0$. In this approach, the stable site is still stable even if this site receives more than one particle. It means that the multitoppling process of one site is omitted in PVZ's RG approaches. Since PVZ used height configuration H and toppling rules configuration E to describe the RG sandpile evolution, the height probability vector $\vec{\mathbf{n}}$ can be simplified to one parameter ρ, which is the probability of critical sites, and $(\vec{\mathbf{n}}^{(k)},\vec{\mathbf{p}}^{(k)})$ becomes $(\rho^{(k)},\vec{\mathbf{p}}^{(k)})$.

An avalanche event specifies how a toppling process goes on. Figure 3 shows a typical avalanche event of a sandpile evolution on a three-site cell of the PT lattice. The labels of sites and toppling directions are shown in Fig. 3(a). Figure 3(b) shows a starting height configuration with site 1 being critical, site 2 being critical, and site 3 being stable, i.e.,

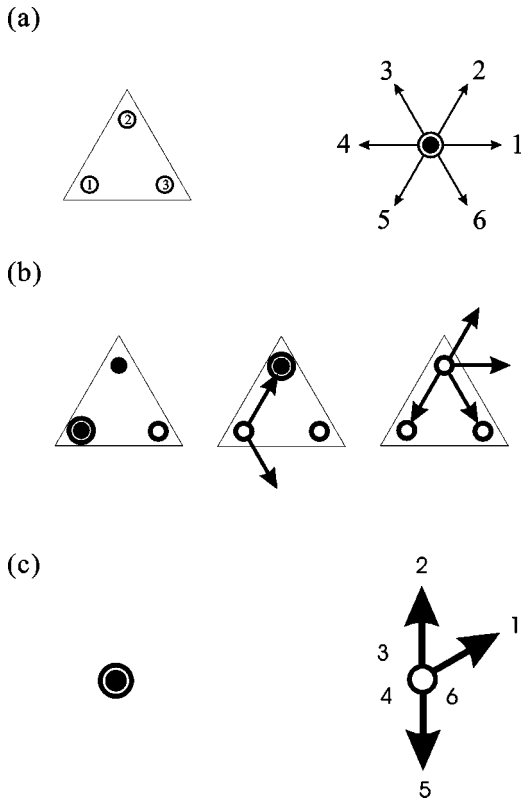

FIG. 3. A toppling process on a three-site cell of the plane triangular lattice. Open dots represent stable sites, black dots represent critical sites, and encircled black dots represent unstable site. (a) The label of sites inside a cell and the label of toppling directions of an unstable site. (b) A series of toppling process from left to right. For site 1 and 2, their toppling rule $\vec{\mathbf{r}}_1$ and $\vec{\mathbf{r}}_2$ are (0, 1, 0, 0, 0, 1) and (1, 1 ,0, 0, 1, 1), respectively. (c) After the RG transformation, the three-site cell of (a) the toppling processes of (b) are represented a supersite and the toppling rule $\vec{\mathbf{r}}=(1,1,0,0,1,0)$ on the supersite. Note that the toppling directions in (c) can be obtained from (b) by rotating 30° counterclockwise.

$\vec{h}=(h_1,h_2,h_3)=(1,1,0)$. This starting height configuration appears with a probability $\rho^2(1-\rho)$. When we add one particle to site 1, site 1 becomes unstable and transfers one particle to site 2 via directions 2 and out of cell via direction 6. Note that the probability of adding one particle to site 1 is 1/3 since there are three sites inside the RG cell and the probability of transferring particles to directions 2 and 6 from site 1 is p_2/C_2^6 since there are C_2^6 possible toppling rules for transferring two particles to six directions. After site 2 receives one particle from site 1 it becomes unstable and transfers one particle to site one via direction 5, one particle to site 3 via direction 6, and two particles out of cell via directions 1 and 2; the probability of transferring particles to four NN sites from site 2 via directions 1, 2, 5, and 6 is p_4/C_4^6. Since site 1 will not topple again and site 3 is a stable site, the toppling process stops. In summary, the probability of the avalanche event and the relaxation time (t) of Fig. 3(b) are $(1/3)\rho^2(1-\rho)(1/15)p_2(1/15)p_4$ and 2 (we set $t_w=1$), respectively.

Under RG transformation, each three-site cell of $\mathfrak{R}^{(k-1)}$ transfers into one supersite of $\mathfrak{R}^{(k)}$ as shown in Figs. 1(b) and 3(c); a site of $\mathfrak{R}^{(k-1)}$ can transfer particles to six directions shown in the right-hand side of Fig. 3(a) and a supersite of $\mathfrak{R}^{(k)}$ can transfer particles to six directions shown in the right-hand side of Fig. 3(c), which correspond to six outgoing directions of Fig. 1(d). In the avalanche event of Fig. 3(b), the particles which leave the cell in directions 6, 1, and 2 contribute to outgoing particles in directions 5, 1, and 2, respectively, in the figure at the right-hand side of Fig. 3(c) [see also Fig. 1(d)]. Thus the avalanche event of Fig. 3(b) in $\mathfrak{R}^{(k-1)}$ contributes to $p_3^{(k)}$ for the supersite of $\mathfrak{R}^{(k)}$.

In general, the probability of one avalanche event is determined by the initial height configuration, the starting site of the avalanche, and the detail of the toppling process. If the initial height configuration is the jth element of H and contains m' critical sites, the starting site is site α, the toppling process consists of a_k unstable sites which transfer k particles to k different directions for $1\leqslant k\leqslant q$, the probability of this avalanche event is

$$P_E(j,\alpha,a_1,a_2,\ldots,a_q) = W_j W_{\langle\alpha\rangle}\left(\frac{1}{C_1^q}p_1\right)^{a_1}\left(\frac{1}{C_2^q}p_2\right)^{a_2}\cdots\left(\frac{1}{C_q^q}p_q\right)^{a_q}, \quad (1)$$

where $W_j=\rho^{m'}(1-\rho)^{m-m'}$ and $W_{\langle\alpha\rangle}=1/m$, $a_1,a_2,\ldots,$ and a_q are zero or positive integer and $\Sigma_{i=1}^q a_i=m''$ which is the total number of toppled sites. Since we do not consider multiple toppling events (i.e., one site can topple at most once), m'' should be equal to or smaller than m.

To construct RGT, we should first collect all possible events on a RG cell of $\mathfrak{R}^{(k-1)}$. Consider the set of events $C_j(\alpha,a_1,a_2,\ldots,a_q,t)$ which are initiated by the jth initial height configuration of H and starting site α, have relaxation time t, topple a_k times by toppling rule with k directions. For the set $C_j(\alpha,a_1,a_2,\ldots,a_q,t)$, we count the number of events on $\mathfrak{R}^{(k-1)}$ which topple i directions in point of view of site on $\mathfrak{R}^{(k)}$. This number is denoted by $B_i(j,\alpha,a_1,a_2,\ldots,a_q,t)$. Therefore, for $1\leqslant i\leqslant q$

$$\begin{aligned} f_{ij} &= \sum_\alpha \sum_{a_1,a_2,\ldots,a_q} \sum_t B_i(j,\alpha,a_1,a_2,\ldots,a_q,t) \\ &\quad \times W_{\langle\alpha\rangle}\left(\frac{1}{C_1^q}p_1^{(k-1)}\right)^{a_1}\left(\frac{1}{C_2^q}p_2^{(k-1)}\right)^{a_2}\cdots\left(\frac{1}{C_q^q}p_q^{(k-1)}\right)^{a_q} \\ &= \sum_\alpha \sum_{a_1,a_2,\ldots,a_q} \sum_t B_i(j,\alpha,a_1,a_2,\ldots,a_q,t) \\ &\quad \times P_E(j,\alpha,a_1,a_2,\ldots,a_q)/W_j \end{aligned} \quad (2)$$

is the summation of probability of toppling events on $\mathfrak{R}^{(k-1)}$ which evolve under the jth initial configuration on $\mathfrak{R}^{(k-1)}$ and contribute to the $p_i^{(k)}$ on $\mathfrak{R}^{(k)}$. Note that it is possible to have toppling processes inside a cell but transfer nothing outside a cell. PVZ omits all of these events and Ivashkevich *et al.* [31] simply assumed these events appear with probabil-

TABLE I. Critical density and transferring probabilities at the RG fixed point.

	ρ^*	p_1^*	p_2^*	p_3^*	p_4^*	p_5^*	p_6^*
SQ [2×2] [a]	0.515	0.327	0.437	0.205	0.031		
SQ [2×2] [b]	0.468	0.240	0.442	0.261	0.057		
SQ [3×3] [a]	0.663	0.572	0.353	0.070	0.005		
PT [3 sites] [a]	0.214	0.000002	0.0005	0.040	0.314	0.582	0.073
HC [6 sites] [a]	0.763	0.702	0.285	0.013			

[a]This work based on PVZ approach.
[b]PVZ [14].

ity p_0. Now the properties of the system are fully characterized by the distribution $(\rho^{(k)},\vec{\mathbf{p}}^{(k)})$ at this scale. Then, the RG equation can be written as

$$\begin{pmatrix} p_1^{(k)} \\ p_2^{(k)} \\ \vdots \\ p_q^{(k)} \end{pmatrix} = W_1/N_w \begin{pmatrix} f_{11}/F_1 \\ f_{21}/F_1 \\ \vdots \\ f_{q1}/F_1 \end{pmatrix} + W_2/N_w \begin{pmatrix} f_{12}/F_2 \\ f_{22}/F_2 \\ \vdots \\ f_{q2}/F_2 \end{pmatrix} + \cdots + W_n/N_w \begin{pmatrix} f_{1n}/F_n \\ f_{2n}/F_n \\ \vdots \\ f_{qn}/F_n \end{pmatrix}, \tag{3}$$

where n is the total number of configurations in H which could contribute to the right-hand side of Eq. (3), W_j, the weight of the jth element of H, is a function of $\rho^{(k-1)}$, N_w and F_j are the normalized constants of W_j and f_{ij} with $N_w = \Sigma_j W_j$ and $F_j = \Sigma_i f_{ij}$, respectively. This RG equation can be well understood by the transition rate and master equation of sandpile evolution [27,31].

Another important assumption is the inflow of particles equals the flow of particles out of the system, i.e., in the stationary state, $\partial/\partial t \langle \rho \rangle = 0$ [14,31]. This implies $\rho^{(k-1)}$ is a function of $p_j^{(k-1)}$ for j from 1 to q and can be written as

$$\rho^{(k-1)} = \frac{1}{\sum_i i \times p_i^{(k-1)}}. \tag{4}$$

Replacing $\rho^{(k-1)}$ in W_j and N_w of Eq. (3) by Eq. (4), $p_i^{(k)}$ is a pure function of $p_j^{(k-1)}$ for j from 1 to q. From RG assumption, at $k \to \infty$, the critical transferring probability $\vec{\mathbf{p}}^* = (p_1^*, p_2^*, \ldots, p_q^*)$ and the critical density ρ^* can be obtained.

III. CALCULATIONS OF CRITICAL EXPONENTS

In the following calculations, we use the RG approach in the form of Eqs. (3) and (4). First, we consider the RG calculation from the small-scale transformation for square and triangle lattices which are shown in Fig. 1. All bonds outgoing from a given cell into another are included into one renormalized bond of the supersite, as is shown in Figs. 1(c) and 1(d). It is one of the simplest choices where supersites of Figs. 1(c) and 1(d) contain just a few sites of Figs. 1(a) and 1(b), respectively. A block of sites in $\Re^{(k-1)}$ (i.e., an RG cell) is replaced by a site of $\Re^{(k)}$. Here, g is equal to 2 and $\sqrt{3}$ for square and triangle lattices, respectively. A more complex choice for the square lattice can be found in [30] where one supersite contains five sites. In the present paper, for the square lattice we will extend this study to the case where one RG cell contains nine sites. This is the largest RG cell which has been considered for the RG approach to the BTW sandpile model.

By using the algorithm of appendix, we enumerate all possible toppling events for two kinds of cells shown in Fig. 1. However, as in [14,15] we drop those events in which the number of toppled sites are smaller than $g_t = 2$; here it is assumed that such events contribute to p_0 which has been discussed in Sec. II. In other words, these events are assumed to transfer no particle to nearest-neighbor cells. Of course, this is an approximation. From the procedures stated above, we can express W_j and f_{ij} of Eq. (3) in term of $\vec{\mathbf{p}}^{(k-1)}$ and $\rho^{(k-1)}$. The normalized factors W_n and F_i can also be calculated. By using initial values of $\vec{\mathbf{p}}^{(0)}$ and $\rho^{(0)}$ to iterate Eqs. (3) and (4) until $k \to \infty$, we can obtain the fixed point of $\vec{\mathbf{p}}$ and ρ shown in Table I. Note that our results for the 2×2 cell to one-site RG transformation for the square lattice are slightly different from those of Refs. [14,15]. This is due to the difference in values of $A_j(i, a_1, a_2, a_3, a_4)$ discussed in the Appendix. For each lattice, we choose some different starting points of $\rho^{(0)}$ and $\vec{\mathbf{p}}^{(0)}$. We find that all starting points evolve into the same fixed point which is shown in Fig. 4. This shows that there is only one attractive fixed point of the RG equation for square and triangle lattices in this RG transformation.

In [14], the critical exponent of the avalanche distribution τ in $P(s) \sim s^{-\tau}$ is calculated at the fixed point of the RG transformation through the fixed point parameters ρ^* and $\vec{\mathbf{p}}^*$. First, the probability that one avalanche occurs on $\Re^{(k-1)}$ but doesn't occur on $\Re^{(k)}$ can be written as

$$K = \sum_{i=1}^{q} p_i^* (1-\rho^*)^i. \tag{5}$$

In addition, the probability K can also be understood as the probability that the linear dimension of the avalanche l is larger than ag^{k-1} (the lattice constant of $\Re^{(k-1)}$) and smaller than ag^k (the lattice constant of $\Re^{(k)}$). Thus, $P(s)ds=P(l)dl\sim s^{-\tau}ds=l^{1-2\tau}dl$, where the toppling area $s\sim l^2$, and we have

$$K=\frac{\int_{ag^{k-1}}^{ag^k}P(l)dl}{\int_{ag^{k-1}}^{\infty}P(l)dl}=1-g^{2(1-\tau)} \tag{6}$$

also holds. Using Eqs. (5) and (6), we obtain τ which is listed in part (A) of Table II. We find that our τ for square and triangle lattices are close to the numerical simulations [28,32] for the square lattice.

Another independent critical exponent is the dynamical exponent z. From the scaling laws at the fixed point we know that the average time of a dynamical process scales with the linear length as $\langle t\rangle\sim l^z$. Therefore, the time scale t_k of a relaxation event on lattice $\Re^{(k)}$ and t_{k-1} on the lattice $\Re^{(k-1)}$ are related by the relation $t_k/t_{k-1}=(ag^{(k)}/ag^{(k-1)})^z=g^z$. On the other hand, the time scale t_k can be obtained as a function of the time scale t_{k-1} from the RG equations. The relation is given by $t_k=\langle t^{(k-1)}\rangle t_{k-1}$, and

$$\langle t^{(k-1)}\rangle=\sum_{i,j,\alpha,a_1,a_2,\ldots,a_q,t_{k-1}}\frac{W_i}{N_w}\frac{W_\alpha B_j(i,\alpha,a_1,a_2,\ldots,a_q,t_{k-1})}{F_i}\times\left(\frac{1}{C_1^q}p_1^{(k-1)}\right)^{a_1}\left(\frac{1}{C_2^q}p_2^{(k-1)}\right)^{a_2}\cdots\left(\frac{1}{C_q^q}p_q^{(k-1)}\right)^{a_q}(t_{k-1}), \tag{7}$$

where $\langle t^{(k-1)}\rangle$ is the average number of subprocesses on $\Re^{(k-1)}$ needed to have a relaxation process on $\Re^{(k)}$. By inserting the fixed point parameters into the calculation of $\langle t\rangle=\lim_{k\to\infty}\langle t^{(k)}\rangle$, we obtain the following result for the dynamical exponent [14]:

$$z=\frac{\ln\langle t\rangle}{\ln(g)}. \tag{8}$$

Using Eqs. (7) and (8), we obtain z which is listed in part (A) of Table II. We find that our result $z=1.284$ for the triangle lattice is not far from the theoretical prediction value $z=1.25$ [19]. If the universality is valid for square and triangle lattices, the obtained z must be the same both lattices. In our RG calculations, we find z for the square lattice has a larger deviation from the theoretical predicted value: $z=1.25$.

We also consider the 3×3 cell for the square lattice, which is shown in Fig. 5. Here $g_t=3$. The critical density of sites and transferring probabilities are shown in Table I. We find that there are some difference between the ρ^* and p_i^* in the 2×2 cell to one site and the 3×3 cell to one-site RG transformations. The exponents of τ and z shown in Table II have larger deviations from the simulation and theoretical prediction results than the results obtained from the 2×2 cell.

IV. REAL-HEIGHT RENORMALIZATION-GROUP EQUATIONS

In the above study, the height configuration Z is simply characterized by H. In this section, the real-height configuration Z is used to build the RG equation. Instead of $(\rho,\vec{\mathbf{p}})$ in the above calculations, $(\vec{\mathbf{n}},\vec{\mathbf{p}})$ is used in this section, where $\vec{\mathbf{n}}=(n_0,n_1,\ldots,n_{q-1})$ with $n_0+n_1+\cdots+n_{q-1}=1$ and n_i is the probability that the height is i. Therefore, the W_i of Eq. (1) is replaced by $W_i=\Pi_{j=1}^m n_{z_j}$. And the steady-state equation of Eq. (4) is now characterized by the following conditions: $\dot{n}_0=\dot{n}_1=\cdots=\dot{n}_{q-1}=0$ [16]. Therefore, the relationships between concentration of height $\vec{\mathbf{n}}^{(k)}$ and transferring probability $\vec{\mathbf{p}}^{(k)}$ at the stationary state is

$$n_{i-1}^{(k)}=\left(\sum_{j=1}^{i}p_{q-j}^{(k)}\right)/\overline{p^{(k)}}, \tag{9}$$

where $\overline{p^{(k)}}=\Pi_{i=1}^q i\times p_i^{(k)}$ is the average number of particles sent from one site to other sites.

By using the computer algorithm in the Appendix, we count all events over $Z\otimes\Psi\otimes R$. Here, for height configuration $(z_1,z_2,\ldots,z_m)$, there are q possible states for each site, i.e., $z_i=0,1,\ldots,$ or $q-1$. Totally, q^m possible configurations are considered. Therefore, the number of events with real height Z is much larger than that with H. According to Eqs. (3) and (9), the p_i^* and n_i^* can be obtained after repeated iterations. In Table III, we compare the critical height probabilities with the exact [10] and numerical results. The numerical results are obtained from simulations on 1000×1000 SQ and PT lattices and in each case 10^6 configurations are generated to obtain the data. We also list the results of RG calculations reported in Refs. [16,17]. We find that our RG fixed point is very close to previous RG calculations [16,17]. It means that our RG calculation is reliable. We also

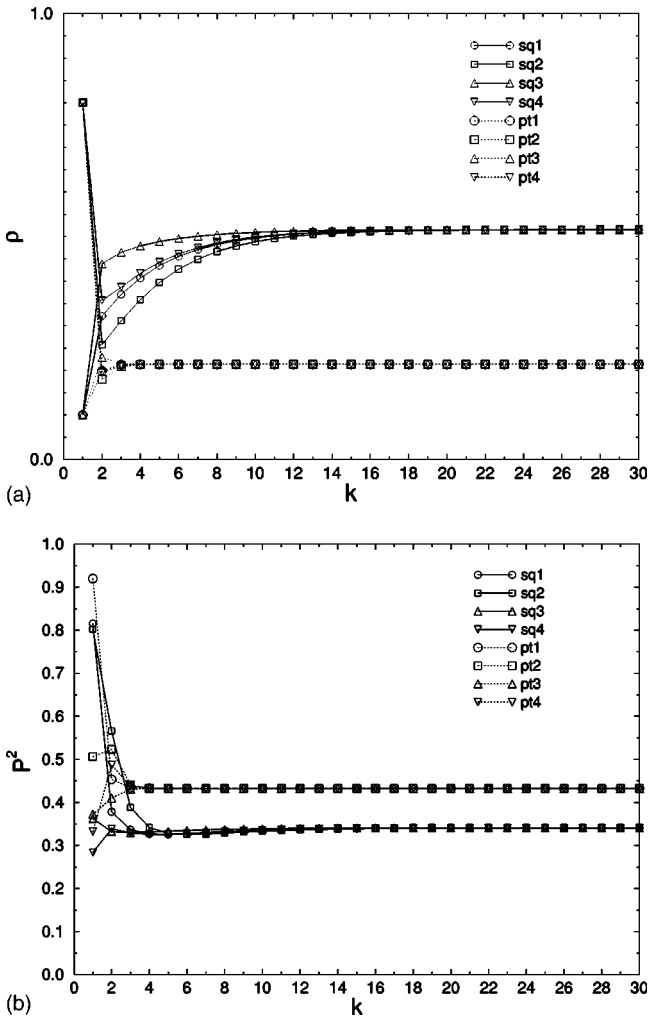

FIG. 4. Iteration results for different initial values of $\vec{\mathbf{p}}^{(0)}$ and $\rho^{(0)}$. The solid line with symbols ○, □, △, and ▽ represent the 2×2 cell transformation on square lattice with initial values $\vec{\mathbf{p}}^{(0)}=(0,0,0,1),(0,0,0,1)$, $(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})$, and $(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})$ and $\rho^{(0)}=0.1,0.8,0.1$, and 0.8, respectively. The dashed line with symbol ○, □, △, and ▽ represent the three-site cell transformation on triangle lattice with initial values $\vec{\mathbf{p}}^{(0)}=(0,0,0,0,0,1)$, $(0,0,0,0,0,1)$, $(\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6})$, and $(\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6})$ $\rho^{(0)}=0.1,0.8,0.1$, and 0.8, respectively. (a) A plot of the $\rho^{(k)}$ against the kth iteration number k. (b) A plot of the $|\vec{\mathbf{p}}^{(k)}|^2=(p_1^{(k)})^2+(p_2^{(k)})^2+\cdots+(p_q^{(k)})^2$ against the kth iteration number k.

find that the RG results of the critical height probabilities n_i^* have the same behavior as the numerical or exact results.

For critical exponent τ, Eq. (6) is still useful but Eq. (5) should be revised to satisfy the definition of $\vec{\mathbf{n}}$. The new equation is

$$K=\sum_{i=1}^{q} p_i^*(1-n_q^*)^i, \tag{10}$$

where ρ^* in Eq. (5) has to be replaced by n_q^*. By Eqs. (6) and (10), the critical exponent τ is obtained for SQ and triangle lattice which are shown in part (B) of Table II. We find that the value of τ is very close to previous calculation for SQ and triangle lattice. It shows again that our computer algorithm and calculation are equivalent to Ivashkevich's algorithm. In Refs. [16,17], they did not calculate the dynamical exponent z. Here, by Eqs. (7) and (8), we calculate this quality which is shown in part (B) of Table II. We find that the obtained z on the triangle lattice is far from the value of exact and numerical results for the square lattice.

TABLE II. Avalanche exponent τ and dynamical exponent z for square and plane triangular lattices.

(A) Reduced parameter	τ (SQ)	z (SQ)	τ (PT)	z (PT)
SQ [2×2] [a]	1.243	1.147		
SQ [2×2] [b]	1.253	1.168		
SQ [5 sites] [c]	1.235	1.236		
SQ [3×3] [a]	1.122	1.082		
PT [3 sites] [a]			1.363	1.284
(B) Real parameter				
SQ [2×2] [a]	1.248	1.150		
SQ [2×2] [d]	1.248			
PT [3 sites] [a]			1.367	1.433
PT [3 sites] [e]			1.367	
Simulation	1.33 [f]	1.254 [i]		
Prediction	1.25 [g] and 1.2 [h]	1.25 [i]		

[a]This work
[b]PVZ [14].
[c]Moreno, Gomez, and Pacheco [30].
[d]Ivashkevich [16].
[e]Papoyan and Povolotsky [17].
[f]Lubek and Usadel [28].
[g]Priezzhev, Ktitarev, and Ivashkevich [20].
[h]Tebaldi, Menech, and Stella [21].
[i]Majumdar and Dhar [19].

V. CALCULATIONS FOR HONEYCOMB LATTICE

There is no overlap of sites between RG cells in the Figs. 1 and 5. In other words, from these two figures, if one site belongs to one specified cell, this site does not belong to other cells. Due to this property of cells, Eqs. (5)–(10) can be used to calculate critical exponents. Consider the RG transformation cell for the honeycomb (HC) lattice shown in Fig. 6. This kind of RG cell allows one site to belong to two different RG cells. If we use the cell transformation in Figs. 6(a) and 6(b), we can still obtain the fixed point of ρ and p_i. However, it seems that it is inappropriate to get the critical exponents τ and z for the HC lattic by Eqs. (5)–(10).

The algorithm in the Appendix is still useful for the calculation of transferring probabilities for the honeycomb lattice. First, for the reduced-height RG equation, we count all possible toppling events and drop those events with $g_t=3$. Then, using Eqs. (3) and (4) to obtain ρ^* and p_i^* which are

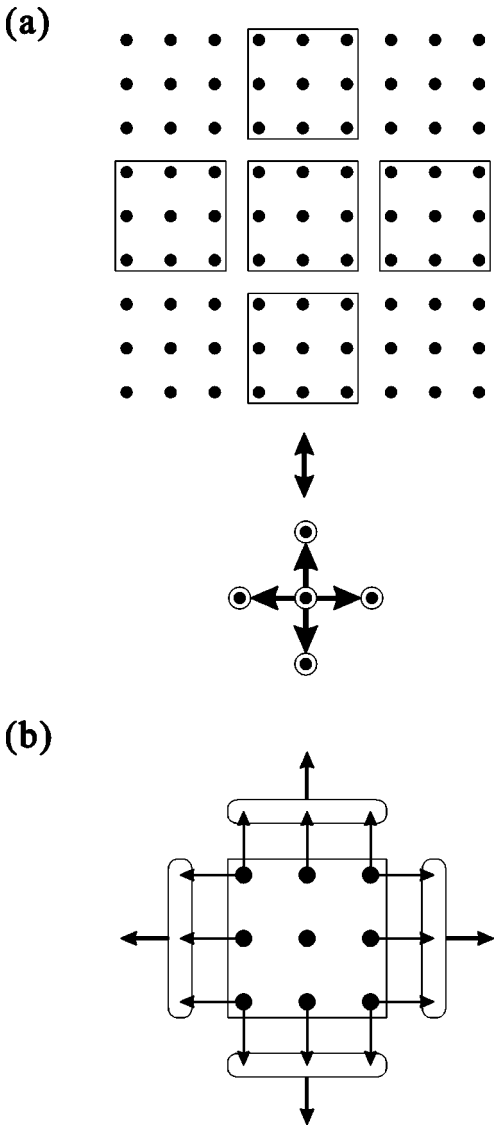

FIG. 5. Transformation from a cell with nine sites on the square lattice. (a) This shows the transformation from a cell to a site. (b) We show that the directions outgoing from the blocks, which are encircled by rectangle corner roundness, are coupled to the directions of the lattice at the next scale.

shown in Table I. For real-height RG equation, the obtained n_i^* and p_i^* are shown in Tables III and IV, respectively. We find that the height probabilities n_i are quite consistent with the numerical simulations on 1000×1000 honeycomb lattice, where 10^6 configurations are generated to obtain numerical data.

The branching probability of a spanning tree [9] $\hat{p}_k$ is the probability for any site of a random spanning tree to have a coordination number k. The Green's function of the Laplace equation for the square, triangle, and honeycomb lattices are well known [33]. For an infinite lattice, G depends only on the difference of site coordinates rather than their values. Here, we show the Green's function for the honeycomb lattice as an example:

TABLE III. Comparisons of critical height probabilities n_i^* for square (SQ), plane triangular (PT), and honeycomb (HC) lattices obtained by RG transformation, numerical simulations, and exact calculation. In the numerical simulations with statistics of 10^6 configurations on 1000×1000 lattices are generated to obtain the data.

	n_0^*	n_1^*	n_2^*	n_3^*	n_4^*	n_5^*
RG (SQ) [a]	0.021	0.134	0.349	0.496		
RG (SQ) [b]	0.021	0.134	0.349	0.496		
Simulation (SQ) [a]	0.074	0.174	0.306	0.446		
Exact (SQ) [c]	0.074	0.174	0.306	0.446		
RG (PT) [a]	0.036	0.135	0.198	0.210	0.211	0.211
RG (PT) [d]	0.036	0.135	0.198	0.210	0.211	0.211
Simulation (PT) [a]	0.058	0.094	0.139	0.188	0.240	0.281
RG (HC) [a]	0.014	0.308	0.678			
Simulation (HC) [a]	0.083	0.293	0.624			

[a]This work based on real height RG approach.
[b]Ivashkevich [16].
[c]Priezzhev [10].
[d]Papoyan and Povolotsky [17].

$$G(r_1-r_2) = \int_0^{2\pi}\int_0^{2\pi}\frac{1}{2}\frac{f(x_1,x_2,y_1,y_2,\alpha,\beta)}{1-\frac{4}{9}(4\cos^2\alpha 4\cos\alpha\cos\beta)}\frac{d\alpha}{2\pi}\frac{d\beta}{2\pi}, \tag{11}$$

where

$$f(x_1,x_2,y_1,y_2,\alpha,\beta) = \cos[\alpha(x_1-x_2)\beta(y_1-y_2)]^{\frac{2}{3}}\cos(\alpha)\cos[\alpha(x_1-x_2) \times\beta(y_1-y_2)]^{\frac{1}{3}}\cos(\alpha(x_1-x_2)\beta(y_1-y_2)\beta)-1. \tag{12}$$

Following the same procedure as Ref. [34], we obtain the branching probability of a spanning tree $\hat{p}_k$ for the honeycomb lattice. The values of $\hat{p}_k$ for SQ and PT lattices [17,34] are also listed in Table IV. In Table IV, we compare the RG transferring probabilities p_i^* with the branching probabilities of spanning tree $\hat{p}_i$. We find that they have the same behavior on the square lattice. However, they are not on the triangle and honeycomb lattices. Thus, we can conclude that the hypothesis about the coincidence of p_i and $\hat{p}_i$ proposed in [16] is not valid.

VI. SUMMARY AND DISCUSSION

In this paper we use a computer algorithm to calculate the effective toppling events for two kinds of RG equations. We

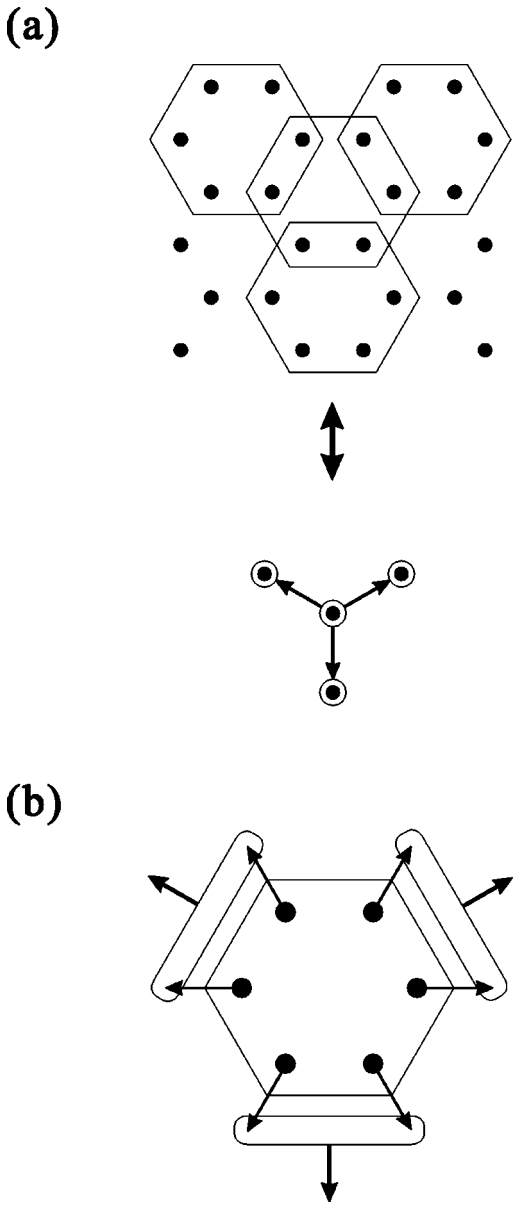

FIG. 6. Transformation from a cell with six sites on the honeycomb lattice (a) This shows a transformation from a cell to a site. (b) We show that the directions outgoing from the blocks, which are encircled by rectangle corner roundness, are coupled to the directions of the lattice at the next scale.

find that the values of critical exponents τ and z for the 2×2 cell to one-site RG transformation for the SQ lattice are closer to conjectured exact values and simulation values than those for the 3×3 cell to one-site RG transformation. There are two possible reasons. The first reason is that we do not consider the multiple toppling events in the RG calculations. The errors arising from the multiple toppling events are larger for the 3×3 cell. Therefore, when the cell size increases, we should consider the multiple toppling events in order to get accurate values of τ and z. This cannot be done easily in exact enumeration approach to RG transformations, but can be done by Monte Carlo RG calculations. Another possible reason is that the critical exponent τ is not well defined for the BTW sandpile model [21] and therefore RGT cannot be used to obtain the critical exponent.

Every avalanche of a BTW sandpile model can be represented as a sequence of more elementary events, called toppling waves [12,13,35]; each toppled site in a wave topples exactly once in that wave. Since the RG transformation (RGT) presented in the present paper does not allow the multiple toppling events, it seems that the RGT can be used to calculate critical behavior of waves. It is well known that the critical exponent of the size distribution of all toppling waves is 1 [12,13,35]. Table II shows that in the reduced-parameter RG calculations for the SQ lattice, when the linear size of the RG cell increases from 3 to 2, τ(SQ) decreases from 1.243 to 1.122 which is very close to 1: the critical exponent for all waves. However, it seems that the RGT can only give approximate critical exponent for all waves because the height configuration after relaxation of a wave is usually still unstable (except the last wave [12,13,36]) and after the RGT, the system has a stable height configuration. It is worth mentioning that as the dimension of the BTW model increases, the multiple toppling events become rare [37]. Under this condition, an avalanche usually only have one wave and the RGT can give better a result.

For the PVZ approach, the critical density of sites ρ^* is equivalent to height probabilities n_q^* in the Ivashkevich approach. If we compare the value of ρ^* with the numerical or exact n_q^*, we find that ρ^* is larger than n_q^*. This is related to the rule that in the PVZ approach, the stable site will not topple even the stable site receive more than one particles. Therefore, the obtained ρ^* must be larger to compensate the loss of stable sites which have the potential of toppling. Again, when the cell size is larger, the compensation effect is also larger. For example, ρ^* obtained from the 3×3 cell of the square lattice is larger than that obtained from the 2×2. There is no such kind of problem in Ivashkevich's approach. We find the n_i^* is consistent with the numerical simulations and exact results.

In summary, it is worthwhile to consider the multiple toppling events in the real-height RG treatment in order to answer the question discussed above. However, it is hard to carry out exact enumerations for larger cell sizes. In the next step, we plan to use Monte Carlo simulations to construct RG transformations with large cells and include multiple toppling events. Such generalization of the RG method could be also valuable to calculate critical exponents for the Manna model [38], which has well-defined avalanche exponents.

ACKNOWLEDGMENTS

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APPENDIX

In lattices $\mathfrak{R}^{(k)}$ for $k\geqslant1$ obtained after RG transformations, one site can transfer particles to one, two, . . . , or q

TABLE IV. Fixed point of transferring probability p_i^* and branching probabilities (BP) of spanning trees $\widehat{p_i}$.

	p_1^* $(\widehat{p_1})$	p_2^* $(\widehat{p_2})$	p_3^* $(\widehat{p_3})$	p_4^* $(\widehat{p_4})$	p_5^* $(\widehat{p_5})$	p_6^* $(\widehat{p_6})$
RG (SQ) [a]	0.295	0.435	0.229	0.041		
RG (SQ) [b]	0.295	0.435	0.229	0.041		
BP (SQ) [c]	0.295	0.447	0.222	0.036		
RG (PT) [a]	0.000 0179	0.002 26	0.0558	0.296	0.471	0.174
RG (PT) [d]	0.000 0179	0.002 26	0.0558	0.296	0.471	0.174
BP (PT) [d]	0.322	0.417	0.207	0.049	0.006	0.0002
RG (HC) [a]	0.546	0.432	0.022			
BP (HC) [a]	0.25	0.5	0.25			

[a]This work based on real height RG approach.
[b]Ivashkevich 1996 [16].
[c]Manna, Dhar, and Majumdar 1992 [34].
[d]Papoyan and Povolotsky 1997 [17].

different directions with probabilities $p_1, p_2, \ldots,$ and p_q, respectively. There are totally q different directions which can be labeled by $d_1, d_2, \ldots,$ and d_q. In a more precise description, a site can transfer one particle or nothing to a specified direction d_i which is denoted by the variable r_{d_i} with values $r_{d_i}=1$ or 0, respectively. Therefore, the toppling rules of sites are exactly determined by the vector $\vec{\mathbf{r}}=(r_{d_1}, r_{d_2}, \ldots, r_{d_q})$. Since one unstable site sends nothing to its neighbors is omitted in PVZ RG transformation, $(r_{d_1}, r_{d_2}, \ldots, r_{d_q})=(0,0,\ldots,0)=\vec{0}$ is forbidden. For a given $\vec{\mathbf{r}}$, the site will send $j=\Sigma_{i=1}^{q} r_{d_i}$ particles to j different directions. If we divide the toppling rule configuration set $\Im=\{(r_{d_1}, r_{d_2}, \ldots, r_{d_q}) | r_{d_i}=0 \text{ or } 1, \forall\ i \text{ and } \Sigma_{i=1}^{q} r_{d_i}\neq 0\}$ into subsets $\Im_j=\{(r_{d_1}, r_{d_2}, \ldots, r_{d_q}) | r_{d_i}=0 \text{ or } 1, \forall\ i \text{ and } \Sigma_{i=1}^{q} r_{d_i}=j\}$ according to the value j, q subsets can be obtained. And it is straightforward to show that the number of elements of the jth subset $\Im_j$ and $\Im$ are C_j^q and $\Sigma_{j=1}^{q} C_j^q = 2^q-1$, respectively.

Now we define Prob($\vec{\mathbf{r}}$) as the probability that an unstable site transfers particles with a given toppling rule $\vec{\mathbf{r}}$. The relation $p_j=\Sigma_{\vec{\mathbf{r}}\in\Im_j}\text{Prob}(\vec{\mathbf{r}})$ can be obtained. On the other hand, we assume that the probability Prob($\vec{\mathbf{r}}$) is the same for all $\vec{\mathbf{r}}\in\Im_j$ because of the isotropy of lattice $\Re$. Therefore, we conclude that $\text{Prob}(\vec{\mathbf{r}})=p_i/C_i^q$ for $\vec{\mathbf{r}}\in\Im_i$. For example, $\text{Prob}[\vec{\mathbf{r}}=(1,0,\ldots,0)]=\text{Prob}[\vec{\mathbf{r}}=(0,1,\ldots,0)]=\cdots=\text{Prob}[\vec{\mathbf{r}}=(0,0,\ldots,1)]=p_1/q$ and $\text{Prob}[\vec{\mathbf{r}}=(1,1,\ldots,0)]=p_2/C_2^q$. In this way, $\mathbf{R}=\{(\vec{\mathbf{r}}_1, \vec{\mathbf{r}}_2, \ldots, \vec{\mathbf{r}}_m) | \vec{\mathbf{r}}_i\in\Im, \forall\ i\}$ can be used to represent the toppling rule configuration of an RG cell; $\mathbf{R}$ contains more information than $E=\{\vec{e}\}$ introduced in Sec. II.

In Sec. II, we presented the PVZ approach to construct RG transformation of Eq. (3) by using avalanche events and we used Fig. 3 as an illustrative example. In this section, we will use the idea of toppling rule configuration $\mathbf{R}$ and a computer algorithm to calculate all terms which contribute to the right-hand side of Eq. (3). Consider a RG cell of m sites, whose initial high configuration is $(h_1, h_2, \ldots, h_m)$ and the toppling rule configuration is $(\vec{\mathbf{r}}_1, \vec{\mathbf{r}}_2, \ldots, \vec{\mathbf{r}}_m)$. If we add one particle to site α, the sandpile evolution starts and finally evolves into one state where all new h_i are stable. In order to count all possible avalanche events, we prepare a Fortran computer program SOCRG to generate such events. In SOCRG, we consider three subconfigurations sets and use $(m+2)$ do loops to generate such subconfigurations

(1) The initial height configuration H: There are two possible states for each site, i.e., $h_i=0$ or 1. Therefore, totally, 2^m possible configurations are considered. However, some of these 2^m configurations will not induce sandpile evolution, for example, $h_i=0$ for all i. In SOCRG, we use the first do loop to generate all possible configurations in H.

(2) The starting point set of sandpile dynamics $\Psi=\{\alpha | \alpha=1,2,\ldots,\text{ or } m\}$, where α denotes the site on which we add one particle: There are m possible positions to add one particle to a RG cell with m sites. In SOCRG, we use the second do loop to generate all possible site α in Ψ. If the chosen site α is not a critical site, SOCRG goes to the next site.

(3) The toppling rule configuration $\mathbf{R}$: There are 2^q-1 possible toppling rules for each site. Therefore, there are $(2^q-1)^m$ possible toppling configurations for an m-site cell. In SOCRG, we use the third to the $(m+2)$-th do-loops to generate such toppling configurations.

For a specified configuration with $\vec{h}^s$, α^s, $(\vec{\mathbf{r}}_1^s, \vec{\mathbf{r}}_2^s, \ldots, \vec{\mathbf{r}}_m^s)$ generated by these $(m+2)$ do loops, we can calculate the toppled vector $\vec{O}=(O_1, O_2, \ldots, O_m)$ where $O_i=1$ or 0 depending on whether site i topples or it does not during the sandpile evolution. For example, Fig. 3 shows an event on a three-site RG cell of the triangle lattice. In this example, (h_1^s, h_2^s, h_3^s) obtained from the first do loop, α^s obtained from the second do loop, $\vec{\mathbf{r}}_1^s$ obtained from the third do loop, and $\vec{\mathbf{r}}_2^s$ obtained from the fourth do loop, are (1, 1, 0), 1, (0, 1, 0, 0, 0, 1), and (1, 1 ,0, 0, 1, 1), respectively. Then,

$\vec{O}=(1,1,0)$ is obtained. The weight of this height configuration is $\rho^2(1-\rho)$. The probability of this event is $1/3\rho^2(1-\rho)\mathrm{Prob}(\vec{\mathbf{r}}_1^s=(0,1,0,0,0,1))\mathrm{Prob}[\vec{\mathbf{r}}_2^s=(1,1,0,0,1,1)]$ and the relaxation time $t=2$. Finally, this event is classified as $\vec{\mathbf{r}}=(1,1,0,0,1,0)$ for the supersite of Fig. 3(c). In this case, since site 3 does not topple ($O_3=0$). The probability of the toppling process depends only on $\vec{h}^s$, α^s, $\vec{\mathbf{r}}_1^s$, and $\vec{\mathbf{r}}_2^s$. The do loop used to generate $\vec{\mathbf{r}}_3^s$ can be skipped quickly (see below).

In general, consider the i_1th element of the first do loop $\vec{h}^{i_1}$, the i_2-th element of the second do loop α^{i_2}, and the j_k-th element of the $(k+2)$-th do loop $\vec{\mathbf{r}}^{j_k}$ for $1\leqslant k\leqslant m$; the combination of such elements is denoted by the $D(i_1,i_2,j_1,\ldots,j_k,\ldots,j_m)$, which is a configuration of the set consists of H,Ψ, and $\mathbf{R}$. This configuration will generate a toppling event with probability

$$P_P(\vec{h}^{i_1},\alpha^{i_2},\{\vec{\mathbf{r}}^{j_k}\})=W_{\vec{h}^{i_1}}W_{\langle\alpha^{i_2}\rangle}\prod_{k=1}^{m}[\mathrm{Prob}(\vec{\mathbf{r}}^{j_k})]^{O_k}. \quad \text{(A1)}$$

Since $O_k=0$ means that site k is not involved in the toppling process, any choice of $\vec{\mathbf{r}}$ in the $(k+2)$-th loop will correspond the same event represented by the configuration $D(i_1,i_2,j_1,\ldots,j_k,\ldots,j_m)$. Define a set $\aleph$ which consists of the configurations: $D(i_1,i_2,J_1,\ldots,J_k,\ldots,J_m)$ with $J_k=j_k$ for $O_k=1$ or J for $O_k=0$, where J is an integer and $1\leqslant J\leqslant 2^{q-1}$. From $\vec{O}$, we can calculate the total number of toppled sites: $m''=O_1+O_2+\cdots+O_m$. There are $(2^q-1)^{m-m''}$ elements in $\aleph$ and every such element of $\aleph$ corresponds to the same event which is generated by the configuration $D(i_1,i_2,j_1,\ldots,j_k,\ldots,j_m)$. In SOCRG, the do loops corresponding to sites with $O_k=0$ are passed through quickly to save the computing time and the following technique is used.

Define an m-dimensional array: $V(j_1,j_2,\ldots,j_m)$, where $1\leqslant j_k\leqslant 2^{q-1}$ for $1\leqslant k\leqslant m$. Immediately after a new height configuration is chosen by the first do loop (say the i_1-th step) and a new starting point is chosen by the second do loop (say the i_2-th step), all elements of the array V are set to one. For each $D(i_1,i_2,j_1,\ldots,j_k,\ldots,j_m)$ configuration generated in the third to the $(m+2)$-th do loop, we first check the value of $V(j_1,j_2,\ldots,j_m)$. If $V(j_1,j_2,\ldots,j_m)=1$, we record this event and obtain a set $\aleph$. Then, we set $V(J_1,J_2,\ldots,J_m)=0$ for $D(i_1,i_2,J_1,J_2,\ldots,J_m)$ in $\aleph$. If $V(j_1,j_2,\ldots,j_m)=0$, we skip this step to the next step under the third do loop to the $(m+2)$-th do loop. In this way, the do loops corresponding to sites with $O_k=0$ can be passed through quickly. Repeat above procedure for different combinations of height configuration and starting point and we finish the calculation of all do loops.

Basically, the form of Eq. (A1) can be transferred to the form of Eq. (1). Therefore, we can construct the RG Eqs. (3) and (4). In order to test this algorithm, we calculate RGT for a 2×2 cell to one site on a square lattice, which is shown in Figs. 1(a) and 1(c) and has been done in details by Vespignani, Zapperi, and Pietronero (VZP) [15]. Define

$$A_i(k,a_1,a_2,a_3,a_4)=\sum_j\sum_\alpha\sum_t B_i(j,\alpha,a_1,a_2,a_3,a_4,t)\times\left(\frac{1}{C_1^4}\right)^{a_1}\left(\frac{1}{C_2^4}\right)^{a_2}\left(\frac{1}{C_3^4}\right)^{a_3}\left(\frac{1}{C_4^4}\right)^{a_4}\delta[k,\phi(j)],$$

where $\phi(j)$ is the number of critical sites of the jth configuration of heights. We find that almost all of our calculated values of $A_i(k,a_1,a_2,a_3,a_4)$ are the same as those which appear in the appendix of [15], except $A_3(3,0,1,1,1)=0.749\,997$, $A_2(4,0,4,0,0)=0.177\,2839$, and $A_4(4,2,0,1,1)=1.460\,250$ in [15]. Our values of $A_3(3,0,1,1,1)$, $A_2(4,0,4,0,0)$, and $A_4(4,2,0,1,1)$ are 1.750 000, 0.117 2839, and 1.406 250, respectively. We find that $A_2(4,0,4,0,0)$ and $A_4(4,2,0,1,1)$ in Ref. [15] are only slightly different from our values. They might be typographical errors in [15]. However, there is an obvious difference between our value and VZP's value for $A_3(3,0,1,1,1)$. We believe that VZP's value is wrong since all other 240 terms of VZP are exactly the same as ours and VZP calculated $A_k(i,a_1,a_2,a_3,a_4)$ by hand and we calculate $A_k(i,a_1,a_2,a_3,a_4)$ by a systematic algorithm.

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